THE GEOMETRY OF SCRAP PAPER: AREA

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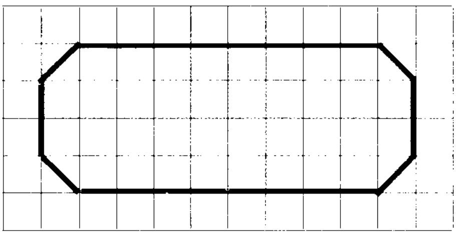
No matter what the shape, students think A = $l \times w$ is the formula to use for finding the area. On Michigan's 7th Grade State Assessment Test, 20% of the students selected the $l \times w$ response of 40 square units on the following test item.

Find the Area of the Shaded Figure in Square Units



These students tried to substitute memorization of formulas for understanding the concept of area.

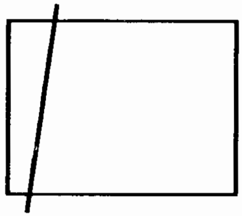
For triangles and quadrilaterals there is an easier way!

All students need to know that finding area means overlaying a square grid and counting the squares. This is best learned by doing. Children should use transparency grids, and/or draw grids, and count the squares to find either the area or its approximation. Drawing grids provides an excellent measurement activity.

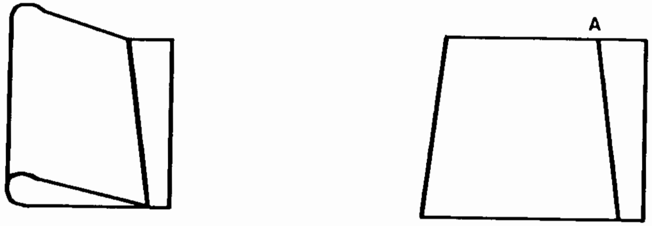
In a very short period of time, students will find that the area of a rectangle is length times width. If other shapes could be rectangularized, finding the area would be easy. Any four-sided figure (quadrilateral) can easily be reshaped into a rectangle.

The Leaning Rectangle: The Parallelogram

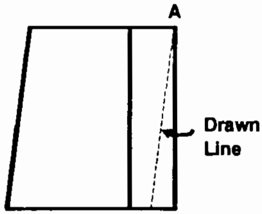
Activity: Tear or cut off one end of a piece of scrap paper (8.5 × 11).



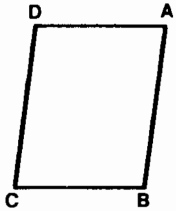
Using the larger piece, carefully roll (do not crease) the paper and draw along the torn edge.



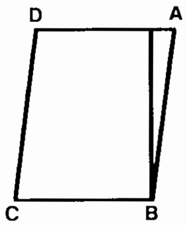
Fold the paper back to form the perpendicular at A.



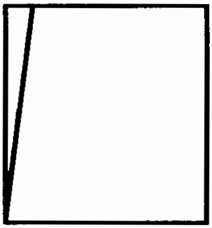
With the paper folded, cut along the line which was drawn. The resulting quadrilateral looks like a tired rectangle, which has leaned to the side. This is a parallelogram!



Fold the perpendicular to line AD through B.

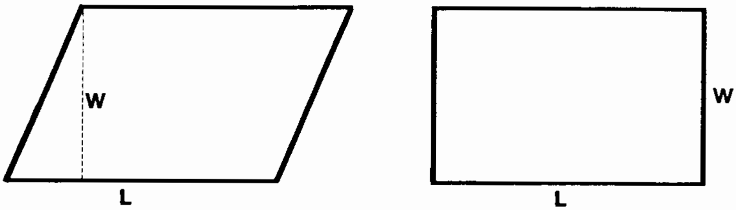


Cut along the fold and reposition the piece at the other end to form a rectangle.



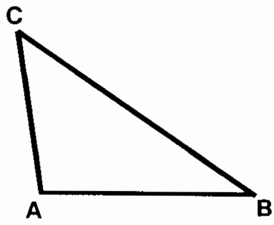
Notice that the length of the rectangle is the same as the length of the parallelogram and the width (height) has not changed.

Therefore, in the following drawing, the area equals length times width, for both figures.

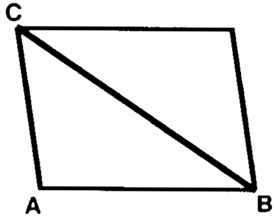


Triangle Activity:

Cut two triangles the same size and shape from scrap paper and stack one on the other.



Rotate the top triangle around the midpoint of line BC.

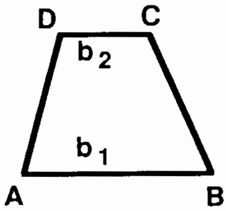


The resulting parallelogram is twice as large as the original triangle. If the area of the parallelogram is length (base) times height, then the area of the triangle is half the base times the height.

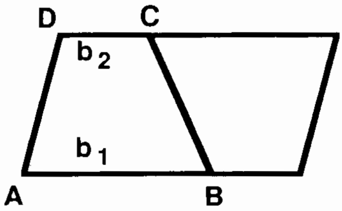
Students should have to draw the parallelogram when finding the area of a triangle.

Trapezoid Activity:

A trapezoid has four sides, two of which are parallel. Take two pieces of scrap paper. With the pieces stacked, fold two opposite ends so they are not parallel. Cut along the folds and the resulting figures are identical trapezoids.



Rotate the top trapezoid a half turn around the midpoint of BC.



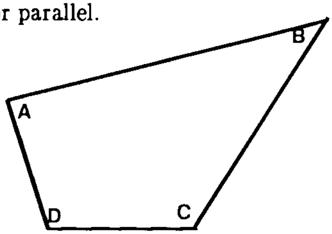
The resulting figure is a parallelogram twice the size of the original trapezoid. The area of the parallelogram is the sum of the bases times the height $A = (b_1 + b_2)h$. Half of this yields the area of the trapezoid.

Students should have to draw the parallelogram when finding the area of the trapezoid.

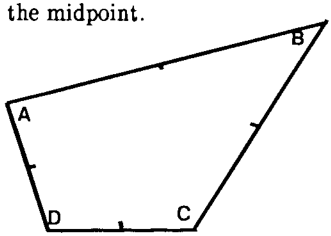
The following activity has many intriguing results, not all of which are directly related to finding area. They are of such interest, however, that they will be investigated.

General Quadrilateral Activity:

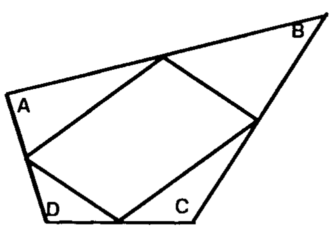
Fold and cut a piece of scrap paper so that it has four sides, no two of which are either perpendicular or parallel.



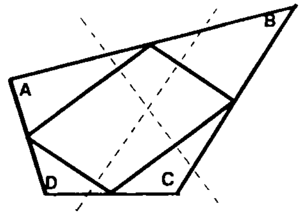
Crease each side at the midpoint.



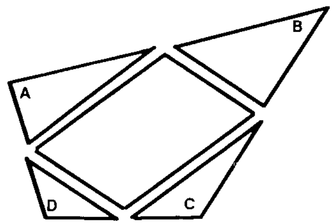
Draw lines connecting the midpoints of adjacent sides.



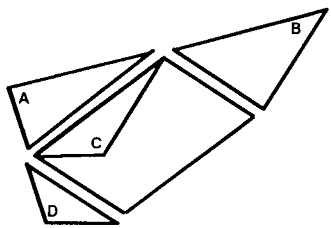
The drawn figure is a parallelogram. The opposite sides can be shown to be parallel by folding a perpendicular to one side and showing that it is also perpendicular to the opposite side. (This could be a paper-folding definition for parallel.)



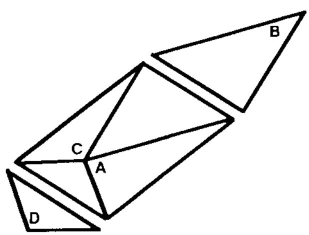
Cut around the parallelogram.



Slide the "C triangle" across the parallelogram and place it on top of the parallelogram with the matching cut edges stacked.

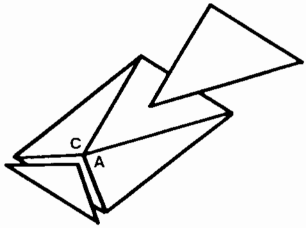


Slide the "A triangle" across the parallelogram and place it on top of the parallelogram in the same manner.

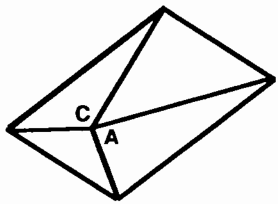


Notice that the A and C vertices are at the same place on the parallelogram. (Their coincidence can be proven.) Two new parallelograms are formed. "D triangle" and the adjacent uncovered portion of the underlying parallelogram form a new parallelogram. Similarly, the "B triangle" and the other uncovered portion form a parallelogram.

Rotate the "B triangle" and the "D triangle".

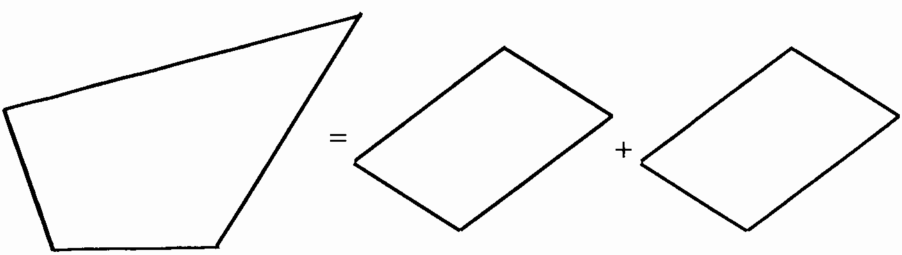


Slide the triangles into position. The triangular pieces exactly cover the parallelogram.



The original quadrilateral is twice as large as the parallelogram. Therefore, by finding the area of the parallelogram and doubling, we obtain the area of the quadrilateral.*

Pictorially,



* This property of quadrilaterals was originally shared with me by Professor James Riley of the Mathematics Department, Western Michigan University. He had discovered it as a high school student. Although this was probably not the first discovery of the property, it certainly was a wonderful experience for a budding mathematician.

Wayne Scott's last maneuver is intriguing. Why does it work? Readers are invited to submit explanations or to offer alternative transformations.

The Editors

Dear Editors,

Just a note to assure your more alarmed teachers that the Standards aren't uniformly impossible to achieve. Take this one for example:

Theorems for circles involving segments should receive less attention.

The other day I heard about a geometry prof at a nearby college who started the course by reviewing high school theorems. In response to "Who can give us a circle theorem?", there was nothing. The usual wait time passed and the prof sketched a circle and a tangent line. Eventually a student said, "Something about a radius being perpendicular to a tangent." Two or three students mumbled agreement, but no improvement was offered. That's all! No one remembered anything else about circles!

Looks like we are right on target with this standard.

Veteran Kid Watcher